Archimedes and π

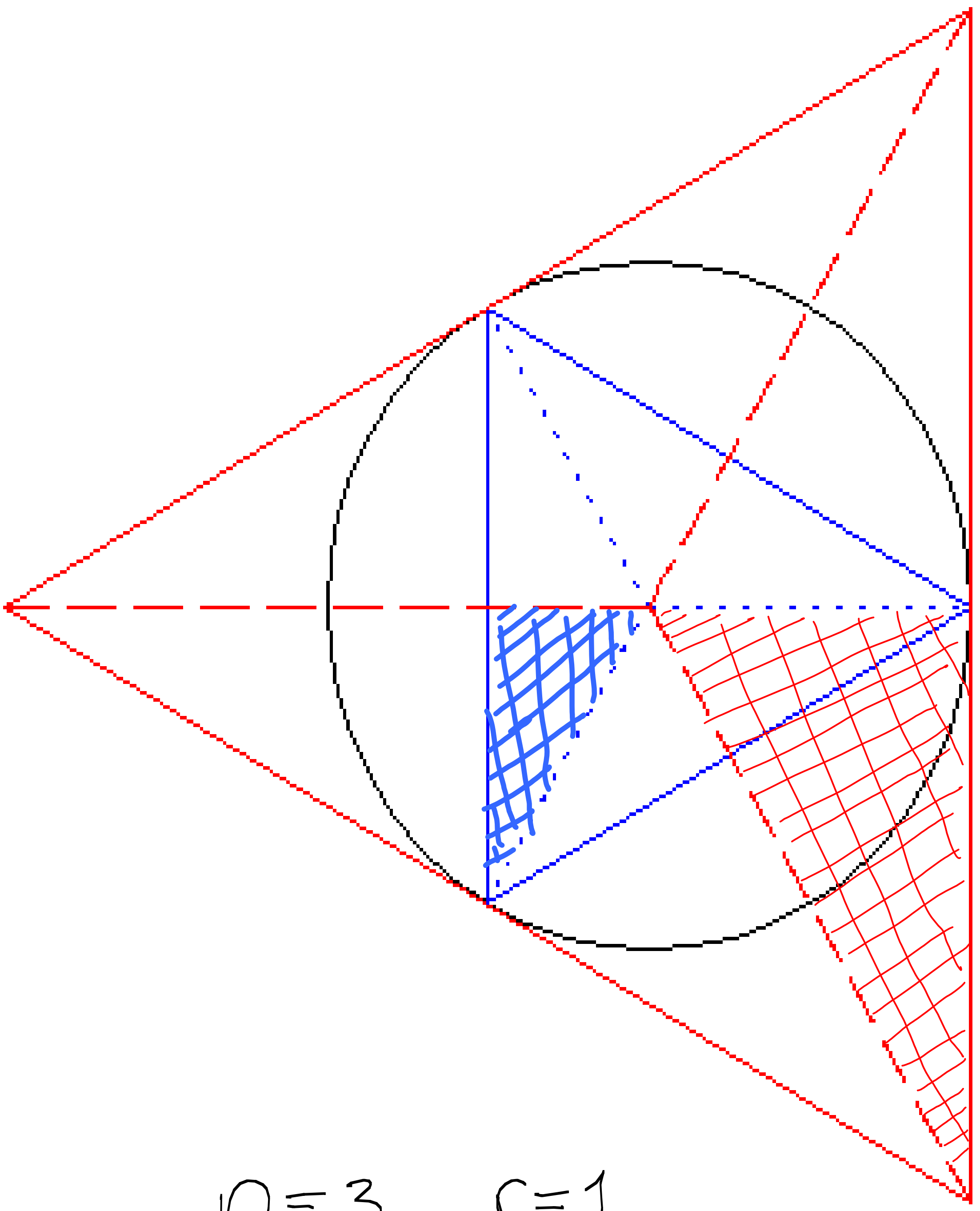
Note Title

30/10/2008

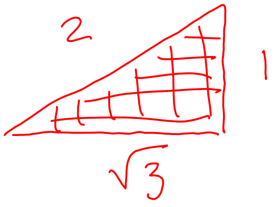
In class today I sketched how Archimedes calculated the area of a circle.

He showed an inscribed polygon had less area than the circumscribed one, and that as n (the number of sides) went to ∞, the inscribed area increased and the circumscribed area decreased, squeezing the "true" area of the circle between them.

These notes record the formulas of today, plus better diagrams.

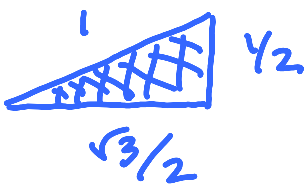


$n=3 \quad r=1.$

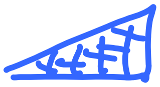


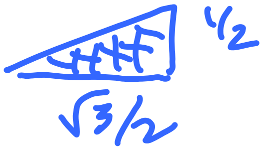
"30-60-90" triangle

$\frac{\pi}{6} - \frac{\pi}{3} - \frac{\pi}{2}$ in radians



similar but smaller

Area of inscribed "3-gon" (triangle) $= 6$ 

Each 

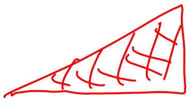
has area $\frac{1}{2} \cdot \text{height} \cdot \text{base}$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}.$$

$$\therefore \ a_3 = \frac{6\sqrt{3}}{8} \doteq 1.299$$

small!

Area of circumscribed "3-gon"

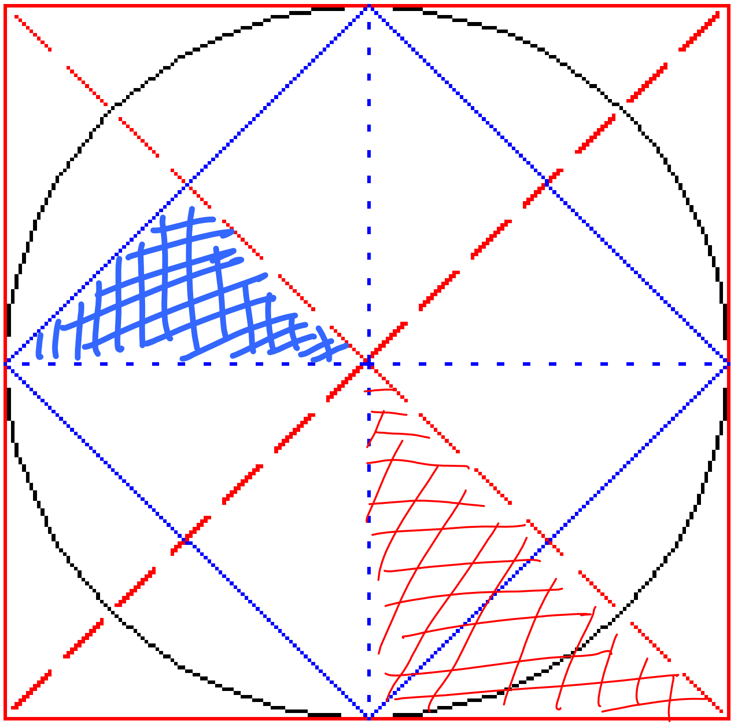
$= 6$ 

$$= 6 \cdot \frac{1}{2} \cdot 1 \cdot \sqrt{3} = 3\sqrt{3} \doteq 5.19$$

too large!

$$1.299 < \pi < 5.19$$

pretty sad



$n = 4$

Area ◇ = 8 △ $1/\sqrt{2}$ $1/\sqrt{2}$ 1

45-45-90

$\frac{\pi}{4}$ - $\frac{\pi}{4}$ - $\frac{\pi}{2}$ Δ

Area □ = 8 △ 1 $\sqrt{2}$ 1

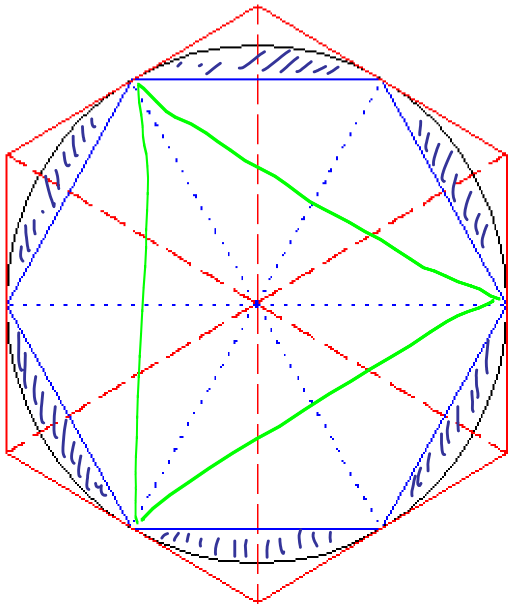
45-45-90 again

$$\therefore \; a_4 = 8 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{8}{4} = 2$$

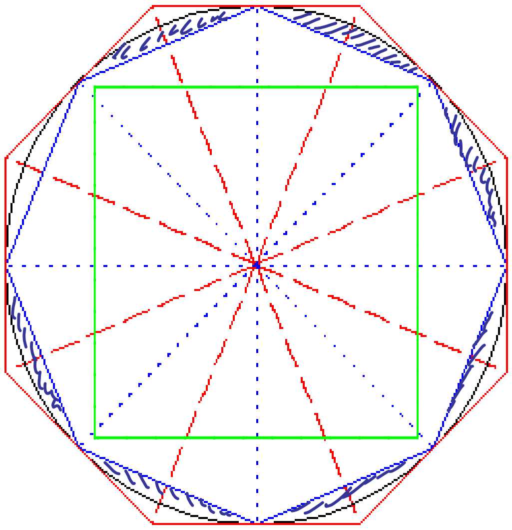
$$A_4 = 8 \cdot \frac{1}{2} \cdot 1 \cdot 1 = \frac{8}{2} = 4$$

$\therefore \quad 2 < \pi < 4$ "wow"

Double # sides



$n = 6$



$n = 8$

area blue < area circle < area red

$$a_n < A_{\odot} < A_n$$

We can walk in Archimedes' footsteps and try to find doubling formulas. First, let $\theta_n = 2\pi/n$ and show that

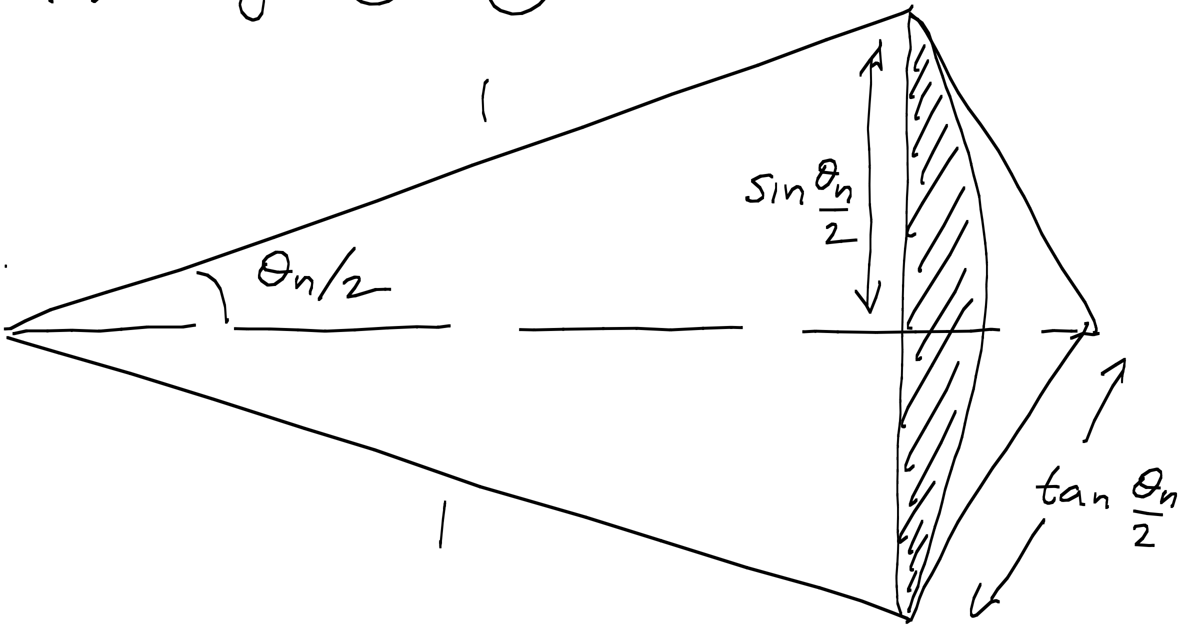
① $a_n = \frac{n}{2}\sin\theta_n = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)$

② $A_n = n\tan\left(\frac{\pi}{n}\right) = n\tan\frac{\theta_n}{2}$

③ $\sin\frac{\theta}{2} = \frac{\sin\theta}{\sqrt{2(1+\cos\theta)}}, \quad \cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$

and $\tan\theta/2 = \frac{\sin\theta}{1+\cos\theta}$

Hint for ① & ②



④ Now combine ①-③ to find formulas for a_{2n} in terms of a_n and A_{2n} in terms of A_n : eg

$$a_{2n} = \frac{2n}{2} \sin\left(\frac{2\pi}{2n}\right) = n \sin\left(\frac{\theta_n}{2}\right)$$

$$= n \frac{\sin\theta_n}{\sqrt{2(1+\cos\theta_n)}}$$

$$= \frac{n \sin\theta_n}{\sqrt{2(1+\sqrt{1-\sin^2\theta_n})}}$$

and since $\sin\theta_n = \frac{2a_n}{n}$,

$$a_{2n} = \frac{2a_n}{\sqrt{2(1+\sqrt{1-(\frac{2a_n}{n})^2})}}$$

∴ If we can take square roots, we can double sides.

Show in this way that

$$A_{2n} = \frac{2A_n}{1+\sqrt{1+\left(\frac{A_n}{n}\right)^2}}$$

and that

$$a_{2n} = \frac{A_n}{\sqrt{1+\left(\frac{A_n}{n}\right)^2}} .$$

Solution:

$$A_n = n \tan\left(\frac{\pi}{n}\right)$$

$$A_{2n} = 2n \tan\left(\frac{\pi}{2n}\right)$$

$$= 2n \frac{\sin(\pi/n)}{1 + \cos(\pi/n)}$$

$$= 2n \frac{\tan \pi/n}{\sec \pi/n + 1}$$

$$= \frac{2n \tan \pi/n}{1 + \sqrt{1 + \tan^2 \pi/n}}$$

$$= \frac{2 A_n}{1 + \sqrt{1 + \left(\frac{A_n}{n}\right)^2}} \qquad 4$$

Likewise

$$a_{2n} = n \sin \frac{\theta_n}{2}$$

$$= n \tan \frac{\theta_n}{2} \cos \frac{\theta_n}{2}$$

$$= \frac{n \tan \frac{\theta_n}{2}}{\sec \frac{\theta_n}{2}}$$

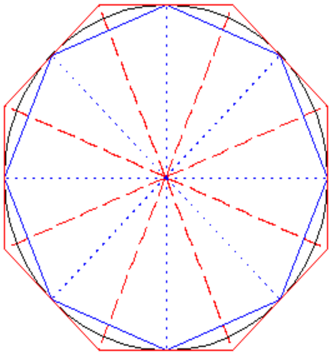
$$= \frac{n \tan \theta_n/2}{\sqrt{1 + \tan^2 \theta_n/2}} = \frac{A_n}{\sqrt{1 + \left(\frac{A_n}{n}\right)^2}}$$

4.

Now let's use these formulas. From a_4 and A_4 we generate a_8 and A_8:

$$a_8 = 2\sqrt{2}$$

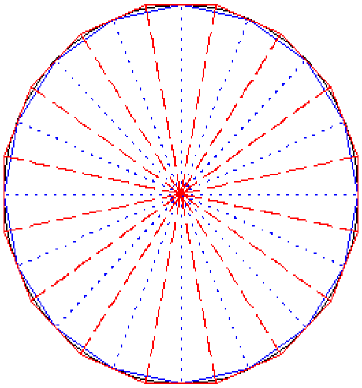
$$A_8 = \frac{8}{1+\sqrt{2}}$$



From these we generate a_{16} and A_{16}

$$a_{16} = \frac{4\sqrt{2}}{\sqrt{2+\sqrt{2}}} \doteq 3.06$$

$$A_{16} = \frac{16}{1+\sqrt{2}+\sqrt{2}\sqrt{2+\sqrt{2}}}$$



$$a_{32} = \frac{8\sqrt{2}}{\sqrt{2+\sqrt{2}}\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

curious pattern

$$A_{32} = \frac{32}{1+\sqrt{2}+\sqrt{2}\sqrt{2+\sqrt{2}}+\sqrt{2+\sqrt{2}}\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

Archimedes invented a way of taking square roots, too. Here we cheat & use a computer: (sure and I've been using it all along ☺)

n $\qquad a_n$ $\qquad A_n$ $\qquad A_n - a_n$

```
4.      2.             4.             2.
8.      2.8284271253.313708499.4852813742
16.     3.0614674593.182597878.1211304192
32.     3.1214451523.151724907.3027975517e-1
64.     3.1365484913.144118385.7569894700e-2
128.    3.1403311573.142223630.1892472988e-2
256.    3.1412772513.141750369.4731182362e-3
512.    3.1415138013.141632081.1182795589e-3
1024.   3.1415729403.141602510.2956988972e-4
2048.   3.1415877253.141595118.7392472429e-5
4096.   3.1415914223.141593270.1848118107e-5
8192.   3.1415923463.141592808.4620295268e-6
16384.  3.1415925773.141592692.1155073817e-6
32768.  3.1415926343.141592663.2887684543e-7
65536.  3.1415926493.141592656.7219211357e-8
```

π accurate to

in between these numbers.

Archimedes got here →

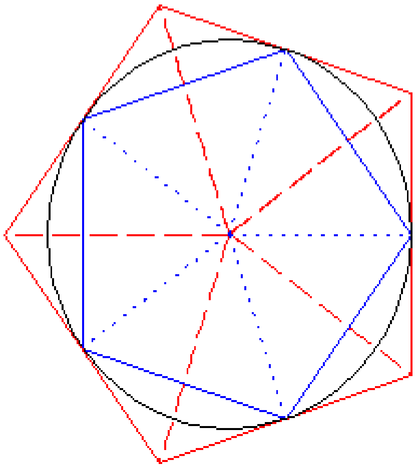
n	a_n	A_n	$A_n - a_n$
3.	1.299038106	5.196152423	3.897114317
6.	2.598076211	3.464101615	.8660254038
12.	3.	3.215390309	.2153903092
24.	3.105828541	3.159659942	.5383140087e-1
48.	3.132628613	3.146086215	.1345760185e-1
96.	3.139350203	3.142714600	.3364396599e-2
192.	3.141031951	3.141873050	.8410990893e-3
384.	3.141452472	3.141662747	.2102747714e-3
768.	3.141557608	3.141610177	.5256869283e-4
1536.	3.141583892	3.141597034	.1314217321e-4
3072.	3.141590463	3.141593749	.3285543302e-5
6144.	3.141592106	3.141592927	.8213858255e-6
12288.	3.141592517	3.141592722	.2053464564e-6
24576.	3.141592619	3.141592671	.5133661409e-7
49152.	3.141592645	3.141592658	.1283415352e-7
98304.	3.141592651	3.141592655	.3208538381e-8
196608.	3.141592653	3.141592654	.8021345952e-9
393216.	3.141592653	3.141592654	.2005336488e-9

These computations nowadays are pretty routine. Imagine doing all those square roots by hand, though.

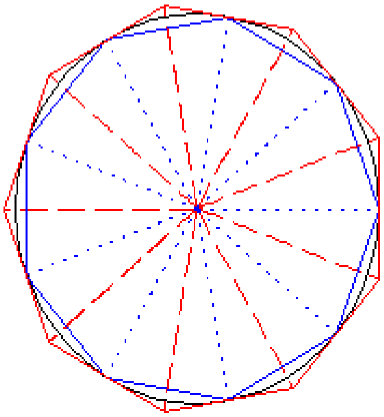
Imagine *inventing* a method for square roots, to do it.

BTW this is the oldest algorithm for π
it is *not* the best.

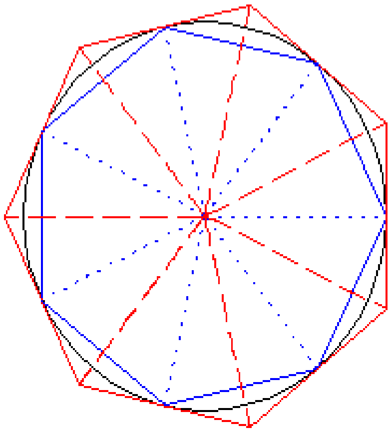
Just for fun



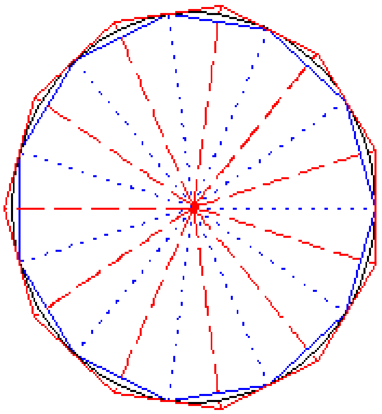
n = 5



n = 9



n = 7



n = 11

These odd ones are tough to start...

In The End

Key observation

By dividing area of a curved figure into a sum of simple pieces, each of which we could find the area of, and by doing this in two ways with a squeeze

$$a_n < a_{2n} < \pi < A_{2n} < A_n$$

and $A_n - a_n \to 0$ as $n \to \infty$

we can define the true area as the common limit $\lim_{n\to\infty} a_n = \lim_{n\to\infty} A_n$.